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LETTER TO THE EDITOR

# Some realizations of the quantum algebra $U_q(su(2))$

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**Abstract.** For the quantum algebra $U_q(su(2))$, a $q$-analogue of the usual spin coherent state is constructed. With help of coherent states the $q$-deformed Dyson and Holstein–Primakoff realizations of the quantum algebra $U_q(su(2))$ are given. A transformation matrix, which turns the Dyson mapping onto the Holstein–Primakoff, is presented.

Over the past couple of years a great deal of attention has been paid to the quantum algebras, especially $U_q(su(2))$ which is a $q$-deformation of the usual Lie algebra su(2). The $U_q(su(2))$ is mathematically a special Hopf algebra which was called a quasitriangular Hopf algebra by Drinfeld [1]. Originally it appeared in studying the properties of the Yang–Baxter equations which play a crucial role in the exactly solvable models in statistical mechanics, and so on. Recently, a $q$-analogue of the Jordan–Schwinger realization for $U_q(su(2))$ has been carried out by many authors [2–5]. In this mapping two kinds of $q$-boson, $a_i^\dagger$ $(a_i)$ with $i=1,2$, must be introduced. In the present letter, we will outline a new realization of the quantum algebra $U_q(su(2))$ which is a $q$-analogue of the usual spin coherent state realization [6–8]. In the new realization, only a single kind of $q$-boson is necessary. Although the idea of the single $q$-boson realization has occurred in the case of $U_q(su(1,1))$ [9–11] our technique is completely different from those works.

The $U_q(su(2))$ is generated by $J_\pm, J_0$ satisfying relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0]. \tag{1}$$

The irreducible representation basis vectors are $|jm\rangle$, and satisfy

$$J_0|jm\rangle = m|jm\rangle \tag{2}$$

$$J_\pm|jm\rangle = \sqrt{[j\mp m][j\pm m+1]}\,|jm\rangle \tag{3}$$

with

$$[x] = (q^x - q^{-x})/(q-q^{-1}) \tag{4}$$

where $q$ is not a root of unity. In the $q\to 1$ the $U_q(su(2))$ reproduce the usual Lie algebra su(2).

We now define a non-normalized $q$-spin coherent state [6–8]

$$|Z\rangle = e_q(Z^* J_-)|jj\rangle \tag{5}$$

where the $|jj\rangle$ is a highest-weight state and satisfy

$$J_+|jj\rangle = 0. \tag{6}$$

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The $e_q(x)$ is a $q$-exponential function ([12] and references therein)

$$e_q(x)=\sum_n x^n/[n]!. \tag{7}$$

Using the following identity

$$[2j]+[2j-2]+[2j-4]+\ldots+[2j-n+2]=[n][2j-n+1] \tag{8}$$

the coherent state (5) can be written as

$$|Z\rangle=\sum_{n=0}^{2j}(Z^*)^n\sqrt{[2j]!/[n]![2j-n]!}\,|j,j-n\rangle. \tag{5'}$$

Here

$$|j,j-n\rangle=\sqrt{[2j-n]!/[2j]![n]!}\,J_-^n|jj\rangle. \tag{9}$$

In analogy to the usual spin coherent states, there exists a resolution of unity for the $q$-spin coherent states (5). The identity operator $I$ can be written as

$$I=\int|Z\rangle\langle Z|\,\mathrm{d}_q\mu(Z) \tag{10}$$

where $\mathrm{d}_q\mu(Z)$ is the $q$-spin coherent-state measure, and defined by

$$\mathrm{d}_q\mu(Z)=\frac{[2j+1]}{2\pi}(1+|Z|^2)^{-2j-2}\mathrm{d}_q(|Z|^2)\,\mathrm{d}\theta. \tag{11}$$

Note that the integral over $\theta$ is a normal integration but the integration $|Z|^2$ is a $q$-integration. The $q$-integration is an inverse operation of the $q$-differentiation ([12] and references therein) which is defined as

$$\frac{\mathrm{d}}{\mathrm{d}_qx}f(x)=\frac{f(qx)-f(q^{-1}x)}{qx-q^{-1}x}. \tag{12}$$

Using $q$-integration by parts, we obtain

$$\int x^n(1+x)^{-m}\,\mathrm{d}_qx=\frac{q^{-n}[n]}{[m-1]}\int x^{n-1}(1+q^{-1}x)^{-m+1}\,\mathrm{d}_qx$$
$$=\frac{q^{-n}[n]q^0}{[m-1]}\frac{q^{-n+1}[n-1]q}{[m-2]}\cdots\frac{q^{-1}[1]q^{n-1}}{[m-n]}\int(1+q^{-n}x)^{-m+n}\,\mathrm{d}_qx.$$

Since

$$\int(1+q^{-n}x)^{-m+n}\,\mathrm{d}_qx=q^n/[m-n-1] \tag{13}$$

we see that all the $q$'s cancel to leave

$$\int x^n(1+x)^{-m}\,\mathrm{d}_qx=\frac{[n]![m-n-2]!}{[m-1]!}. \tag{14}$$

This is a $q$-analogue of the usual beta function. Making use of (14) we can prove (10).

As a result of the resolution of unity, an arbitrary state vector can be expressed through its $Z$-space functional realization

$$\psi(Z)=\langle Z|\psi\rangle. \tag{15}$$

This leads to the scalar product

$$\langle\psi_1|\psi_2\rangle=\int \psi_1(Z)^*\psi_2(Z)\,\mathrm{d}_q\mu(Z)\equiv\langle\psi_1(Z)|\psi_2(Z)\rangle. \tag{16}$$

From (15) we immediately have

$$\psi_{jn}(Z)=\langle Z|j,j-n\rangle=\left(\frac{[2j]!}{[n]![2j-n]!}\right)^{1/2}Z^n \tag{17}$$

and

$$\psi_{jj}(Z)=\langle Z|j,j\rangle=1. \tag{18}$$

Using (10) we can also obtain the $q$-spin coherent state realization of an arbitrary operator $O$,

$$|\phi\rangle=O|\psi\rangle\to\phi(Z)=\mathcal{O}\psi(Z) \tag{19a}$$

$$\langle Z|O|Z'\rangle=\mathcal{O}\langle Z|Z'\rangle. \tag{19b}$$

From (19) we immediately obtain

$$\mathcal{J}_+=\partial_q \qquad \mathcal{J}_-=Z[2j-Z\partial] \qquad \mathcal{J}_0=j-Z\partial. \tag{20}$$

From here on, the abbreviated form of the different symbols $\partial=\partial/\partial Z$ $\partial_q=\partial/\partial_q Z$ is used. We can check that the expression (20) is exact, because

$$\mathcal{J}_+\psi_{jn}(Z)=\sqrt{[n][2j-n+1]}\,\psi_{jn-1}(Z) \tag{21a}$$

$$\mathcal{J}_-\psi_{jn}(Z)=\sqrt{[n+1][2j-n]}\,\psi_{jn+1}(Z) \tag{21b}$$

$$\mathcal{J}_0\psi_{jn}(Z)=(j-n)\psi_{jn}(Z). \tag{21c}$$

With respect to the measure (11), we can prove that the $\psi_{jn}(Z)$ are the orthonormal, and the operators $\mathcal{J}_0, \mathcal{J}_\pm$ are Hermitian, $(\mathcal{J}_0)^+=\mathcal{J}_0$, $(\mathcal{J}_\pm)^+=\mathcal{J}_\mp$.

We now turn the measure (11) onto the $q$-Bargmann measure

$$\mathrm{d}_q\mu(Z)_\mathrm{B}=\frac{1}{2\pi}\,e_q(-|Z|^2)\,\mathrm{d}_q|Z|^2\,\mathrm{d}\theta. \tag{22}$$

With respect to the new measure we can prove that the orthonormalizated basis vectors are

$$X_n(Z)=Z^n/\sqrt{[n]!} \tag{23}$$

that the operators $Z$ and $\partial_q$ satisfy

$$(Z)^\dagger=\partial_q \tag{24}$$

and the operators $\mathcal{J}_0$ and $\mathcal{J}_\pm$, therefore, are not Hermitian. The origin of the non-Hermitian relations are the non-orthonormality of the $\psi_{jn}(Z)$ in (17). The non-Hermitian relations, however, are not a fundamental problem and can be restored by introducing a similarity transformation with an operator $K$

$$\bar{\mathcal{J}}_0=K^{-1}\mathcal{J}_0K \tag{25a}$$

$$\bar{\mathcal{J}}_\pm=K^{-1}\mathcal{J}_\pm K \tag{25b}$$

such that $\bar{\mathcal{J}}_0^+ = \bar{\mathcal{J}}_0$ and $\bar{\mathcal{J}}_\pm^+ = \bar{\mathcal{J}}_\mp$. Note that, since $\mathcal{J}_0$ was Hermitian, no change is needed for it. Thus $K$ can be chosen to commute with $\mathcal{J}_0$ and will thus be diagonal for the quantum number $n$. By requiring that

$$\bar{\mathcal{J}}_- = \bar{\mathcal{J}}_+^+ = (K^{-1}\mathcal{J}_+ K)^+ = K^+ Z (K^{-1})^+ = K^{-1}\mathcal{J}_- K$$

and multiplying from the left by $K$ and from the right by $K^+$, we obtain

$$\mathcal{J}_- K^2 = K^2 Z. \tag{26}$$

Here we have supposed that the operator $K$ is Hermitian, $K^+ = K$. Taking matrix elements between the $X_{n+1}(Z)$ on the left and $X_n(Z)$ on the right, we get

$$K_{n+1}/K_n = \sqrt{[2j-n]}.$$

Here matrix element $K_n \equiv K_{n,n}$. Starting with $K_0 = 1$ leads to

$$K_n = \sqrt{[2j]!/[2j-n]!}. \tag{27}$$

Then the matrix elements of $\bar{\mathcal{J}}_0, \bar{\mathcal{J}}_\pm$ are readily given by

$$\langle X_n(Z)|\bar{\mathcal{J}}_0|X_n(Z)\rangle = j - n \tag{28a}$$

$$\langle X_{n+1}(Z)|\bar{\mathcal{J}}_-|X_n(Z)\rangle = (K_{n+1}/K_n)\langle X_{n+1}(Z)|Z|X_n(Z)\rangle = \sqrt{[n+1][2j-n]} \tag{28b}$$

$$\langle X_n(Z)|\bar{\mathcal{J}}_+|X_{n+1}(Z)\rangle = \langle X_{n+1}(Z)|\bar{\mathcal{J}}_-|X_n(Z)\rangle^*. \tag{28c}$$

Accordingly have

$$\bar{\mathcal{J}}_0 = j - N \tag{29a}$$

$$\bar{\mathcal{J}}_+ = \sqrt{[2j-N]}\,\partial_q \tag{29b}$$

$$\bar{\mathcal{J}}_- = Z\sqrt{[2j-N]}. \tag{29c}$$

Here $N = Z\partial$.

Finally we note that $(\partial_q Z - qZ\partial_q)X_n(Z) = q^{-n}X_n(Z)$. Since $X_n(Z)$ is a complete set, we have therefore

$$\partial_q Z - qZ\partial_q = q^{-N}. \tag{30}$$

Both equations (24) and (30) show that we can introduce a set of $q$-bosons $a^+$, $a$, and $N \neq a^+ a$, satisfying relations

$$[N, a^+] = a^+ \qquad [N, a] = -a \tag{31a}$$

$$aa^+ - qa^+a = q^{-N} \tag{31b}$$

such that there are correspondences

$$Z \to a^+ \qquad \partial_q \to a \qquad Z\partial \to N \tag{32}$$

and

$$X_n(Z) \to (a^+)^n|0)\sqrt{[n]!} \tag{33}$$

where $|0)$ is the $q$-boson vacuum state. Obviously

$$\psi_{jj}(Z) = 1 \to |0). \tag{34}$$

Then we can rewrite the operators of (20), (29) as

$$\mathcal{J}_+ \to \mathcal{J}_+^{(D)} = a \tag{35a}$$

$$\mathcal{J}_- \to \mathcal{J}_-^{(D)} = a^+[2j-N] \tag{35b}$$

$$\mathcal{J}_0 \to \mathcal{J}_0^{(D)} = j - N \tag{35c}$$

and

$$\bar{\mathscr{J}}_{+} \to \mathscr{J}_{+}^{(\mathrm{HP})} = \sqrt{[2j-N]}\, a \qquad (36a)$$

$$\bar{\mathscr{J}}_{-} \to \mathscr{J}_{-}^{(\mathrm{HP})} = a^{+}\sqrt{[2j-N]} \qquad (36b)$$

$$\bar{\mathscr{J}}_{0} \to \mathscr{J}_{0}^{(\mathrm{HP})} = j - N \qquad (36c)$$

where $\mathscr{J}_0^{(\mathrm{D})}$, $\mathscr{J}_\pm^{(\mathrm{D})}$ and $\mathscr{J}_0^{(\mathrm{HP})}$, $\mathscr{J}_\pm^{(\mathrm{HP})}$ are the $q$-Dyson and $q$-Holstein–Primakoff realizations of the $\mathrm{U}_q(\mathrm{su}(2))$ algebras, respectively.

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*Note added.* The $q$-analogue of Bargmann space has been treated in great detail by Bracken *et al* [13].

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